UDC 62-50

ON THE OPTIMAL OBSERVATION PROBLEMS

PMM Vol. 37, №5, 1973, pp. 771-786

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(Received January 21, 1973)

We investigate the observation problems for linear systems operating in the presence of noise not accessible to measurement [1 – 3]. The formulation to be considered covers in a single form both the minimax game situations as well as certain cases of a probabilistic description of observable systems. The desired optimal observation operation is formed in two ways: either a priori, when one and the same operation is selected for all possible realizations of the signal being observed, or a posteriori, when the operation is formulated on the basis of the value realized of this signal. The difference of the two observation methods mentioned are compared and discussed. In particular, the classes of functional constraints on the unknown noise and the class of optimal observation operations when the optimal unimprovable result is achieved, are indicated. The relation of the class of operations indicated with the set of linear observation operations is examined. The paper is closely related to the investigations in [4 – 7].

1. A priori and a posteriori observation. We consider the n-vector-valued controlled system

$$dx / dt = A(t)x + B(t)v + f(t), \qquad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

with an r-vector-valued input function $v(t)$. An m-dimensional quantity $y(t)$, accessible to measurement, is realized by the equation

$$dy / dt = G(t)x + F(t)y + C(t)v + H(t)\xi, \qquad t_0 \leqslant t \leqslant \vartheta_1 \leqslant \vartheta \tag{1.2}$$

where $\xi(t)$ is a q-vector-valued noise in the measurement channel. The coefficients of systems (1.1) and (1.2) are assumed continuous and the Lebesgue-integrable function $f(t)$ is assumed known. The functions $v(t)$, $\xi(t)$ belong to the sets

$$V(\cdot) = V(\mu(\cdot)),\ \Xi(\cdot) = \Xi(\nu(\cdot))$$

which depend in a known way on random functions $\mu(t)$, $\nu(t)$ (on the elements of $\mu(\cdot)$, $\nu(\cdot)$) specified on the interval $t_0 \leqslant t \leqslant \vartheta$. The values $v(t)$, $\xi(t)$ themselves are here assumed to be unknown. Let $f(\cdot) = f(t)$, $t \in [t_0, \vartheta]$.

Let us describe the sets $V(\cdot)$, $\Xi(\cdot)$ in detail. We denote $\mu_1 P = \{p : \mu_1^{-1}p \in P\}$. Here $\mu_1 > 0$, P is a convex set containing zero. Let P and Q be convex compacta in $R^{(r)}$, $R^{(q)}$ $(0 \in P,\ 0 \in Q)$. Further, we assume

$$V(\mu(\cdot)) = \{v(\cdot) : v(t) \in \mu(t)P\}, \qquad \Xi(\nu(\cdot)) = \{\xi(\cdot) : \xi(t) \in \nu(t)Q\}$$

Here $\mu(t) = \varphi_1(\omega_1(t))$, $\nu(t) = \varphi_2(\omega_2(t))$, where $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are convex nonnegative functions, $\varphi_1(0) = \varphi_2(0) = 0$; $\omega_1(t)$, $\omega_2(t)$ are independent random processes with known distributions. We note that if functions $\omega_1(t)$, $\omega_2(t)$ are deterministic, then $V(\mu(\cdot))$, $\Xi(\nu(\cdot))$ also define deterministic classes of functions. In the stochastic version we assume that the distributions of $\mu(t)$, $\nu(t)$ are concentrated on the segments $0 \leqslant \mu(t) \leqslant \mu_0(t)$, $0 \leqslant \nu(t) \leqslant \nu_0(t)$, where $\mu_0(t)$, $\nu_0(t)$ are deterministic functions bounded on $[t_0, \vartheta]$.

We consider methods of constructing a mapping (an "observation operation") $\psi(y(\cdot)) = \psi(\cdot) \in R^{(p)}$ estimating a p-vector-valued parameter $\eta(\vartheta_1) = Nx(\vartheta_1)$ of system (1.1) (N is a known matrix). The accuracy of the estimate is characterized by a nonnegative convex functional $\varphi(\chi)$, $\varphi(0) = 0$, where $\chi(\psi, y) = \eta(\vartheta_1) - \psi(\cdot)$.

We can estimate $\eta(\vartheta_1)$ by two methods. The first method consists in choosing the mapping $\psi(\cdot)$ in advance as one and the same for all possible realizations $y(\cdot)$. The parameter $\eta(\vartheta_1)$ is estimated in accordance with the criterion

$$\varepsilon^\circ = \max_{y(\cdot)} M\,[\varepsilon(\mu(\cdot), \nu(\cdot), \psi(\cdot)) \,/\, y = y(\cdot)] = \min_{\psi(\cdot)} \qquad (1.3)$$
$$\varepsilon(\mu(\cdot), \nu(\cdot), \psi(\cdot)) = \max_{v,\xi} \varphi(\chi)$$
$$v(\cdot) \in V(\mu(\cdot)), \quad \xi(\cdot) \in \Xi(\nu(\cdot)), \quad \psi(\cdot) \in \Psi$$

under the condition

$$\chi(\psi(\cdot), y(\cdot)) = (\psi(y(\cdot)) - \eta(\vartheta_1))|_{v\equiv 0,\, \xi\equiv 0 = 0} \qquad (1.4)$$

Here in (1.3) the conditional mean is taken over all $\mu(\cdot)$, $\nu(\cdot)$ consistent with $y(\cdot)$ (the given operation is explained in detail in Sect. 5 below), and the maximum is then computed over all realizations $y(\cdot)$ which are admissible by the system (1.1), (1.2) for all possible $0 \leqslant \mu(t) \leqslant \mu_0(t)$, $0 \leqslant \nu(t) \leqslant \nu_0(t)$. We call relations (1.3), (1.4) the conditions of the problem of a priori observation of parameter $\eta(\vartheta_1)$ of system (1.1) with respect to the signal $y(t)$ of (1.2). Thus, in the indicated formulation all possible realizations $y(\cdot)$ are "played through" in advance, after which ψ° is selected so as to ensure a certain guaranteed result. In particular, if $\mu(\cdot)$, $\nu(\cdot)$ are nonrandom, then (1.3) turns into the following condition:

$$\varepsilon^\circ = \min_{\psi(\cdot)} \varepsilon(\mu(\cdot), \nu(\cdot), \psi(\cdot)), \quad \psi(\cdot) \in \Psi \qquad (1.5)$$

Principles are presented in [4 – 6] for constructing control problems with parameter-constrained trajectories, connected with the problem being discussed by duality relations.

We turn to the second observation method. Suppose that the signal $y(t) = y^*(t)$ has been realized on the interval $[t_0, \vartheta]$. We now construct the mapping (the "observation operation") with due regard to the fact that the realization $y^*(\cdot)$ is already known. Therefore, among the arguments of ψ we also include the function $y^*(\cdot)$. We estimate $\eta(\vartheta_1)$ in accordance with the criterion ($\psi^*(\cdot) = \psi(y(\cdot), y^*(\cdot))$)

$$\varepsilon^* = M\,[\varepsilon^*(\mu(\cdot), \nu(\cdot), \psi^*(\cdot)) \,/\, y = y^*(\cdot)] = \min_{\psi^*} \qquad (1.6)$$
$$\varepsilon^*(\mu(\cdot), \nu(\cdot), \psi^*(\cdot)) = \max_{v,\xi} \varphi(\chi)$$
$$\psi^* \in \Psi^*, \qquad \{v(\cdot), \xi(\cdot)\} \in W(y^*(\cdot), \mu(\cdot), \nu(\cdot))$$

Here Ψ^* is the class of admissible functionals realizing the observation operation; $W^*(y^*(\cdot), \mu(\cdot), \nu(\cdot))$ consists, correspondingly, of those and only those functions

$v(\cdot)$, $\xi(\cdot)$ with values in $V(\mu(\cdot))$, $\Xi(\nu(\cdot))$ which are consistent with signal $y^*(\cdot)$ (i.e. those which together with the vector $y^*(\vartheta)$ and some $x(\vartheta)$ can generate by virtue of (1.1), (1.2) the realization $y^*(\cdot)$). The conditional mean in (1.6) is computed from the a posteriori distribution of the functions $\mu(\cdot)$, $\nu(\cdot)$ for $y = y^*(\cdot)$. Relations (1.6) are called the conditions of the problem of a posteriori observation of parameter $\eta(\vartheta_1)$ of system (1.1) with respect to the signal $y^*(t)$ from (1.2). In particular, if $\mu(\cdot)$, $\nu(\cdot)$ are nonrandom, then (1.6) turns into the following condition:

$$\varepsilon^* = \min_{\psi^*} \varepsilon^*(\mu(\cdot), \nu(\cdot), \psi^*), \qquad \psi^* \in \Psi^* \tag{1.7}$$

Below, the class Ψ of mappings $\psi(\cdot)$ will consist of the linear operations $\psi(y(\cdot)) = \langle w(\cdot), y(\cdot)\rangle$, continuous on some Banach space B containing the set of all possible realizations $y(\cdot)$. Here the rows of the $(m \times p)$-matrix $w(\cdot)$ belong to space B^*. Space B is constructed below. The class Ψ^* of mappings $\psi(y(\cdot), y^*(\cdot))$ is defined with the aid of linear operations of the form

$$\psi^*(\cdot) = \psi(y(\cdot), y^*(\cdot)) = \langle w(\cdot \,/\, y^*(\cdot)), y(\cdot)\rangle$$

where $y(\cdot) \in B$ and the element $w(\cdot \,/\, y^*(\cdot)) \in B^*$ depends upon the realizations $y^*(\cdot)$. We note that the mapping $\psi(y(\cdot), y^*(\cdot))$ from B into $R^{(p)}$ is now not necessarily linear. Additional constraints on $\psi(y(\cdot), y^*(\cdot))$ can, as in the a priori problem, be included in the requirements on $w(\cdot \,/\, y^*(\cdot)) \in W$.

Below we show that the consideration of only the above-described classes Ψ and Ψ^* is justified by the fact that unimprovable estimates of the parameter $\eta(\vartheta_1)$ can be achieved even in these classes (if space B and set W are chosen properly). Everywhere below we assume $\vartheta_1 = \vartheta$. The extension of the results to the case $\vartheta_1 < \vartheta$ is standard.

Note 1.1. The "non-bias condition" (1.4), signifying that $\psi(y(\cdot)) = x(\vartheta)$ precisely for $v \equiv 0$, $\xi \equiv 0$, is, as will be shown below, necessary for optimality in the sense of criterion (1.3) ((1.6) for the operation $\psi(y(\cdot))$ ($\psi(y(\cdot), y^*(\cdot))$).

Note 1.2. From the sense of the problems being considered the set $W^*(y^*(\cdot), \mu(\cdot), \nu(\cdot))$ of possible noise $\{v(\cdot), \xi(\cdot)\}$ compatible with the signal $y^*(\cdot)$ realized, is necessarily nonempty.

2. Solution of the a priori observation problem. Let $X(t, \tau)$, $Y(t, \tau)$ be the normed fundamental matrices of the systems $x^\cdot = Ax$, $y^\cdot = Fy$, respectively. We have

$$y(t) - Y(t, \vartheta) y(\vartheta) + g(t) = \int_t^\vartheta Y(t, \tau)(H(\tau)\xi(\tau) + \tag{2.1}$$

$$C(\tau) v(\tau))\, d\tau + \int_t^\vartheta Z(t, \xi) B(\xi) v(\xi)\, d\xi - Z(t, \vartheta) x(\vartheta), \quad t \in [t_0, \vartheta]$$

$$Z(t, \vartheta) = \int_t^\vartheta Y(t, \xi) G(\xi) X(\xi, \vartheta)\, d\xi, \quad g(t) = \int_t^\vartheta Z(t, \xi) f(\xi)\, d\xi$$

We denote $z(t) = y(t) - Y(t, \vartheta)\, y(\vartheta) + g(t)$. Taking (1.4) into account, after standard calculations we have

$$\chi(\psi(\cdot), y(\cdot)) = \chi(w(\cdot), v(\cdot), \xi(\cdot)) = \langle w(\cdot), z(\cdot)\rangle = \tag{2.2}$$

$$\left\langle \int_{t_0}^{t} w(\tau)(Z(\tau,t)B(t)-Y(\tau,t)C(t))\,d\tau,\ v(t)\right\rangle -$$

$$\left\langle \int_{t_0}^{t} w(\tau)Y(\tau,t)H(t)\,d\tau,\ \xi(t)\right\rangle$$

$$\int_{t_0}^{\vartheta} w(t)Z(t,\vartheta)\,dt = -N \qquad (2.3)$$

The set of solutions of (2.3) in class B^* is denoted as W_H. Further on, we obtain

$$\varepsilon(\mu(\cdot),\ \nu(\cdot),\ \psi(\cdot)) = \varepsilon(\mu(\cdot),\nu(\cdot),w(\cdot)) = \max_{v,\xi}\varphi(\chi(w(\cdot)), \qquad (2.4)$$

$$\nu(\cdot),\xi(\cdot))) = \sup_\alpha \max_{v,\xi}\{\langle\alpha,\ \chi(w(\cdot),\ v(\cdot),\ \xi(\cdot))\rangle - \varphi^*(\alpha)\}$$

$$\alpha \in R^{(p)}$$

$$\varphi^*(\alpha) = \sup_p\{\langle\alpha,\ p\rangle - \varphi(p)\}$$

$$v(\cdot) \in V(\mu(\cdot)),\quad \xi(\cdot) \in \Xi(\nu(\cdot))$$

Here $\varphi^*(\alpha)$ is a convex function adjoint to $\varphi(p)$, $p \in R^{(p)}$ [8]. In particular, if $\varphi(p) = \max_\alpha\langle p,\alpha\rangle$, $\alpha \in A^*$, i.e. $\varphi(p) = \rho(p;\ A^*)$ is the support function of convex set A^*, then (2.4) is transformed to the equality

$$\varepsilon(\mu(\cdot),\nu(\cdot),w(\cdot)) = \max_\alpha \max_{v,\xi}\{\langle\alpha,\chi(w(\cdot),v(\cdot),\xi(\cdot))\rangle\} \qquad (2.5)$$

$$\alpha \in A^*,\ v(\cdot) \in V(\mu(\cdot)),\ \xi(\cdot) \in \Xi(\nu(\cdot))$$

In the general case we have

$$\varepsilon(\mu(\cdot),\nu(\cdot),w(\cdot)) = \sup_\alpha\{\rho(q(t;\ \alpha w(\cdot))H(t);\Xi(\nu(\cdot))) + \qquad (2.6)$$

$$\rho(s(t,\alpha w(\cdot))B(t) - q(t,\alpha w(\cdot))C(t); V(\mu(\cdot)) - \varphi^*(\alpha)\}$$

$$\alpha \in R^{(p)}$$

Here $s(t,\ \alpha w(\cdot))$, $q(t,\ \alpha w(\cdot))$ are, respectively the n- and m-vector-valued solutions of the system

$$s^\cdot = -sA(t) + qG(t),\quad q^\cdot = -qF(t) + \alpha w(t) \qquad (2.7)$$

$$s(t_0) = 0,\quad q(t_0) = 0$$

$\rho(h(\cdot); Q)$ is the support functional of set $Q(\cdot)$, i.e.

$$\rho(h(\cdot);\ Q) = \sup_q\langle h(\cdot),\ q(\cdot)\rangle,\quad q(\cdot) \in Q$$

Finally, we have

$$f(\mu(\cdot),\nu(\cdot),w(\cdot),\alpha) = \left\{\int_{t_0}^{\vartheta}[\nu(t)\rho(q(t,\alpha w(\cdot))H(t);Q) + \right. \qquad (2.8)$$

$$\left.\mu(t)\rho(s(t,\alpha w(\cdot))B(t) - q(t,\alpha w(\cdot))C(t);\ P)]\,dt - \varphi^*(\alpha)\right\}$$

$$\varepsilon(\mu(\cdot),\nu(\cdot)w(\cdot)) = \sup_\alpha f(\mu(\cdot),\nu(\cdot),w(\cdot),\alpha),\quad \alpha \in R^{(p)} \qquad (2.9)$$

$$\varepsilon^\circ = \inf_w \max_{y(\cdot)} M\varepsilon(\mu((\cdot),\nu(\cdot),w(\cdot)),\quad w(\cdot) \in W_0 = W \cap W_H$$

Here, W_H is determined from (2.3), while W is given by the conditions of the problem. If A^* is a unit sphere in the finite-dimensional space X $(A^* = \{\alpha\colon \|\alpha\| \leqslant 1\})$ and, consequently, $\varphi(p) = \|p\|^*$ (the norm of p in the metric of X^*), using (2.8) and the results in Part 3 of [8], we obtain

$$\varepsilon^\circ(\mu(\cdot), \nu(\cdot), w(\cdot)) = \max_\alpha \Big\{\int_{t_0}^{\vartheta} [\nu(t)\rho(q(t, \alpha w(\cdot)) H(t); Q(\cdot)) + \qquad (2.10)$$

$$\mu(t)\rho(s(t, \alpha w(\cdot)) B(t) - q(t, \alpha w(\cdot)) C(t); P]\, dt\Big\} \quad (\|\alpha\| = 1)$$

We note three special cases of the problem

1) Let $p = 1$, $\varphi(l) = |l|$, then

$$\Phi(w(\cdot)) = \max_{y(\cdot)} \quad M\varepsilon(\mu(\cdot), \quad \nu(\cdot), \quad w(\cdot)) =$$
$$f(\mu_0(\cdot), \nu_0(\cdot), w(\cdot), 1), \quad \varphi^*(1) = 0$$

2) Let $\varphi(l) = \|l\|$, $\nu(t) \equiv 0$, $\mu(t) = \mu$ be a random quantity, then

$$\Phi(w(\cdot)) = \max_\alpha f(\mu_0, 0, w(\cdot), \alpha), \qquad \|\alpha\|^* = 1$$

Analogously, if $\mu(t) \equiv 0$, $\nu(t) = \nu$ is a random quantity, then

$$\Phi(w(\cdot)) = \max_\alpha f(0, \nu_0, w(\cdot), \alpha), \quad \|\alpha\|^* = 1$$

3) Let $\mu(t)$, $\nu(t)$ not be random, then

$$\Phi(w(\cdot)) = \varepsilon(\mu(\cdot), \nu(\cdot), w(\cdot))$$

The space B of m-vector-valued functions $z^*(\cdot)$ is chosen either in the form $B = C_k^m[t_0, \vartheta]$, where the index k depends only on the actual structure of system (1.1), (1.2), or in the form $B = L_q^{(m)}$, $q \geqslant 1$. We note that the lower bound in (2.9) is automatically reached if set W is weakly compact in B.

3. Exact a posteriori estimate. Suppose that a realization $y^*(\cdot)$ of signal $y(t)$, observed relative to system (1.1), (1.2), is known. We derive an exact description of the region $X(y^*(\cdot), \mu(\cdot), \nu(\cdot))$ of those vectors x which are consistent with $y^*(\cdot)$ when $v(\cdot) \in V(\mu(\cdot))$, $\xi(\cdot) \in \Xi(\nu(\cdot))$, assuming that the functions $\mu(\cdot)$, $\nu(\cdot)$ are deterministic. In other words, we find all those vectors x^* for each of which we can find a pair $v^1(\cdot) \in V(\mu(\cdot))$, $\xi^1(\cdot) \in \Xi(\nu(\cdot))$ such that (1.1), (1.2) has the function $y(t) = y^*(t)$ as its solution (for $x(\vartheta) = x^*$, $y(\vartheta) = y^*(\vartheta)$, $v = v^1(\cdot)$, $\xi = \xi^1(\cdot)$). We denote

$$T_1 x = f^{(1)}(\cdot), \qquad f^{(1)}(t) = (t)\, Z(t, \vartheta)\, x$$

$$T_2 v(\cdot) = f^{(2)}(\cdot), \quad f^{(2)}(t) = \int_t^{\vartheta} (-Y(t, \sigma) C(\sigma) + Z(t, \sigma) B(\sigma))\, v(\sigma)\, d\sigma$$

$$T_3 \xi(\cdot) = f^3(\cdot), \quad f^{(3)}(t) = \int_t^{\vartheta} (-Y(t, \sigma) H(\sigma))\, \xi(\sigma)\, d\sigma$$

Here T_1, T_2, T_3 are continuous linear operators from $R^{(n)}$, $L_2^{(r)}$, $L_2^{(q)}$, respectively, into $C^{(m)}$, $t_0 \leqslant t \leqslant \vartheta$. According to (2.1) we have

$$z^*(\cdot) = -T_1x + T_2v(\cdot) + T_3\xi(\cdot) \tag{3.1}$$

Here $z^*(\cdot)$ is connected with $y^*(\cdot)$ in the same way as $z(t)$ with $y(t)$, and is determined completely by specifying the pair $\{y^*(\cdot), f(\cdot)\}$. In accordance with this we shall consider the quantity $z^*(\cdot)$ instead of $y^*(\cdot)$ in all the subsequent functional relations. The reverse transition from $z^*(\cdot)$ to $y^*(\cdot)$ is effected in standard fashion. Therefore, we shall not clarify this in what follows.

The set $X(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ consists of those and only those vectors x for which (3.1) is solvable in the class $V(\mu(\cdot))$, $\Xi(\nu(\cdot))$. Following [7, 9], we obtain the following assertion ($\langle\lambda(\cdot), h(\cdot)\rangle$ is a continuous linear functional on B, $h \in B$, $\lambda(\cdot) \in B^*$, T^* is the operator adjoint to T).

Lemma 3.1. In order that $x \in X(z^*(\cdot), \mu(\cdot), \nu(\cdot))$, it is necessary and sufficient that the inequality

$$\langle T^*\lambda(\cdot), x\rangle \leqslant \sup_{v,\xi} \psi(\lambda(\cdot), v(\cdot)\,\xi(\cdot)) \tag{3.2}$$

$$v(\cdot) \in V(\mu(\cdot)),\quad \xi(\cdot) \in \Xi(\nu(\cdot))$$

$$\psi(\lambda(\cdot), v(\cdot), \xi(\cdot)) = \langle T_2^*\lambda(\cdot), v(\cdot)\rangle + \langle T_3^*\lambda(\cdot), \xi(\cdot)\rangle + \langle\lambda(\cdot), z^*(\cdot)\rangle$$

be fulfilled for any $\lambda(\cdot) \in B^*$.

Lemma 3.2. The set $X(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ is convex and closed.

This property follows from formula (3.2).

Let $L = \{l: \exists\, \lambda(\cdot) \in B^*,\ T_1^*\lambda(\cdot) = l,\ l \in R^{(n)}\}$. Set L is a subspace of $R^{(n)}$. From (3.2) we now conclude the validity of the next assertion.

Lemma 3.3. $x \in X(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ if and only if for any $l \in L$

$$\langle l, x\rangle \leqslant \varphi_0(l) \tag{3.3}$$

$$\varphi_0(l) = \inf\{\rho(T_2^*\lambda(\cdot);\ V(\mu(\cdot))) + \rho(T_3^*\lambda(\cdot);\ \Xi(\nu(\cdot))) + \langle\lambda(\cdot), z^*(\cdot)\rangle\} \tag{3.4}$$

over all $\lambda(\cdot) \in \Lambda(l) = \{\lambda(\cdot):\ T_1^*\lambda(\cdot) = l\}$.

We can extend the definition of the function $\varphi_0(l)$ to the set $L_1 = R^{(n)} \setminus L$ by setting $\varphi_0(l) = \infty$, if $l \in L_1$.

Lemma 3.4. The function $\varphi_0(l)$, $l \in R^{(n)}$ is convex and positive-homogeneous.

These properties follow from the definition of $\varphi_0(l)$. Applying the results of Sect. 13 of [8], from formula (3.3) and Lemma 3.4 we conclude

Lemma 3.5. The formula

$$\varphi_0(l) = \rho(l;\ X(z^*(\cdot), \mu(\cdot), \nu(\cdot))), \quad l \in R^{(n)}$$

is valid.

The function $\rho(l;\ X(z^*(\cdot), \mu(\cdot), \nu(\cdot)))$ is uniformly bounded for all

$$l \in S_1(L) = \{l \in L:\ \langle l, l\rangle = 1\}$$

The uniform boundedness of $\rho(l;\ X(z^*(\cdot), \mu(\cdot), \nu(\cdot)))$ on $S_1(L)$ follows from the analogous property for $\varphi_0(l)$. We note that the following representation of the vectors: $x = x^\circ + x^1$ follows from Lemma 3.5; moreover,

$$X\,(z^*(\cdot),\mu(\cdot),\nu(\cdot))=\{x: x^\circ\in X_L\,(z^*(\cdot),\,\mu(\cdot),\,\nu(\cdot)),\ x^1\in L_1\}$$

$$X_L\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot)) = X\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))\ \cap\ L$$

$$L^1 = \{x: \langle x, l\rangle = 0,\ l\in L\}$$

The set $X_L\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ is a convex compactum in $R^{(n)}$ of dimension $n_1 = n - n_2$, where $n_2 = \dim\ L^1$ is the dimension of L^1, $n_1 = \dim L$ is the dimension of L. Note that if $G = \text{const}$, then $\rho\,(l; X\,(z^*\ (\cdot),\ \mu(\cdot),\ \nu(\cdot))) = 0$ when $l'G = 0$ (the prime denotes transposition).

The special case $L = R^{(n)}$ is interesting. It occurs if and only if the equation $T_1^*\ \lambda(\cdot) = l$ is solvable with respect to $\lambda(\cdot)$ for all l. But the last condition exactly expresses the requirement of complete observability of system (1.1), (1.2) on the interval $[t_0,\ \vartheta]$ when $v\equiv 0$, $\xi\equiv 0$ [1 – 3, 10] (or, in other words, the requirement that condition (1.4) be fulfilled if we assume that

$$\psi\,(y\,(\cdot))=\langle\lambda\,(\cdot),\,z\,(\cdot)\rangle\,|_{\xi\equiv 0, v\equiv 0},$$

see Note 1.1). The next assertion is obtained by standard methods of control theory (for example, see [1, 10]).

Lemma 3.6. In order that the equation $T_1^*\ \lambda(\cdot) = l$ be solvable for any $l\in R^{(n)}$ (i.e. that the system (1.1), (1.2), $\xi\equiv 0$, $v\equiv 0$ be completely observable on the interval $[t_0,\ \vartheta]$), it is necessary and sufficient that the form

$$l'\left(\int_{t_0}^{\vartheta} Z'(t,\vartheta)\,Z(t,\vartheta)\,dt\right) l = l'Hl \tag{3.5}$$

be positive definite. In the stationary case system (1.1), (1.2) is completely observable if and only if the rank of the matrix $Q_1 = [D, DA_1, \ldots, DA_1^{n-1}]$, where

$$D=(0,\,E^{(m)}),\qquad A_1=\begin{pmatrix}A & 0\\ G & F\end{pmatrix}$$

equals $n + m$.

Corollary 3.1. In order that the convex and closed set $X\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ be bounded, it is necessary and sufficient that form $l'\ Hl$ be positive definite.

The vectors $l\in L$ are called observable directions.

Let $Z(\cdot) = z^*(\cdot) + T_1X\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$, where T_1X is the image of set X under mapping T_1. By the symbol $W(\cdot) = \{v(\cdot),\ \xi(\cdot)\}$ we denote the preimage of set $Z(\cdot)$ in $L_2^{(r)}\times L_2^{(q)}$ under the mapping $T\,\{v(\cdot),\xi(\cdot)\} = T_2v(\cdot) + T_3\xi(\cdot)$. Let $W^*(\cdot) = W(\cdot)\cap\{V(\mu(\cdot))\times\Xi(\nu(\cdot))\}$. The projections of $W^*(\cdot) = W^*\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ onto $L_2^{(r)}$ and $L_2^{(q)}$ are denoted, respectively, by $V^*\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ and $\Xi^*\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$. The assertion follows from the convexity and closedness of $X\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ and also from the boundedness of the sets $V(\mu(\cdot))$, $\Xi(\nu(\cdot))$.

Lemma 3.7. The set $W^*\,(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ is convex and weakly compact in $L_2^{(r)}\times L_2^{(q)}$.

From the minimax theorem in [11] we obtain

$$\varphi_0(l) = \max\nolimits_{v,\,\xi}\ \inf\nolimits_\lambda\ \psi\,(\lambda(\cdot),\ v(\cdot),\ \xi(\cdot))$$

$$\lambda(\cdot)\in\Lambda(l),\ v(\cdot)\in V(\mu(\cdot)),\quad \xi(\cdot)\in\Xi(\cdot)$$

Lemma 3.8. Let there be given the pair $\{v^1(\cdot),\ \xi^1(\cdot)\}\in W^*\,(z^*(\cdot),\ \mu(\cdot),$

$v(\cdot))$ and the observable direction l; then $\psi(\lambda(\cdot), v^1(\cdot), \xi^1(\cdot)) \equiv k > -\infty$ on the set $\lambda(\cdot) \in \Lambda(l)$.

The boundedness of $\psi(\lambda(\cdot), v^1(\cdot), \xi^1(\cdot))$ over all $\lambda(\cdot) \in \Lambda(l)$ follows from the observability of direction l and can be verified by a direct calculation. The constancy of the functional $\psi(\lambda(\cdot), v^1(\cdot), \xi^1(\cdot))$ on the affine set $\{\lambda(\cdot) \in \Lambda(l)\}$ follows from its linearity in $\lambda(\cdot)$. From Lemmas 3.5, 3.7, 3.8, and from the fact that when $\{v^1(\cdot), \xi^1(\cdot)\} \in W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ we have

$$\inf_{\lambda(\cdot)} \psi(\lambda(\cdot), v^1(\cdot), \xi^1(\cdot)) = \varphi_0(l) = -\infty, \quad \lambda(\cdot) \in \Lambda(l)$$

for any $l \in L$ (the proof of this fact is analogous to the one in [7]), we conclude, using notation (2.7), that the assertion is valid.

Lemma 3.9. Let $l \in L$ (l is an observable direction). Then for arbitrary $\lambda(\cdot) \in \Lambda(l)$ we have

$$\rho(l; X(z^*(\cdot), \mu(\cdot), \nu(\cdot))) = \langle \lambda(\cdot), z^*(\cdot) \rangle + \rho(s(t; \lambda(\cdot)) \cdot B(t) - q(t, \lambda(\cdot)) C(t), -q(t, \lambda(\cdot)) H(t); \tag{3.6}$$

$$W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)))$$

Let us find an exact estimate of parameter $\eta(\vartheta)$ for fixed $\mu(\cdot)$, $\nu(\cdot)$. We consider the set $N^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)) = NX(z^*(\cdot), \mu(\cdot), \nu(\cdot))$. Here $N^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)) = N^*(\cdot)$ is the region of values of $\eta(\vartheta)$, consistent with the signal $z^*(\cdot)$. Let us find the point $\eta^\circ(\vartheta)$ called the "Chebyshev center" of set $N^*(\cdot)$. By definition we have $(\eta \in N^*(\cdot))$

$$\varepsilon^* = \max_\eta \| \eta - \eta_0 \| = \min_\zeta \max_\eta \| \eta - \zeta \|, \quad \zeta \in R^{(p)}$$

Then for deterministic $\mu(\cdot)$, $\nu(\cdot)$ we obtain that ε^* is a solution of problem (1.7) with $\varphi(\chi) = \| \chi \|$. It is clear that $\eta^\circ = \eta^\circ(N^*(\cdot)) = \eta^\circ(N^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)))$. It now remains to set $\psi^\circ(z^*(\cdot), z^*(\cdot)) = \eta^\circ(N^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)))$. As we can convince ourselves (since under the conditions being considered ε^* is an unimprovable a posteriori estimate of the error in the determination of parameter $\eta(\vartheta)$, and this is true by construction!), $\psi^\circ(z^*(\cdot), z^*(\cdot))$ is the solution of problem (1.7). If $\varphi(\chi)$ is an arbitrary nonnegative convex functional, then η° is the so-called "φ-center" of the set $N^*(\cdot)$, i.e. $(\eta \in N^*(\cdot))$

$$\varepsilon^* = \max_\eta \varphi(\eta - \eta^\circ) = \min_\zeta \max_\eta \varphi(\eta - \zeta), \quad \zeta \in R^{(n)} \tag{3.7}$$

Noting that the formula

$$\rho(\alpha; N^*(\cdot)) = \rho(\alpha N; X(z^*(\cdot), \mu(\cdot), \nu(\cdot))) \tag{3.8}$$

is valid, we find $(\alpha, \zeta \in R^{(p)})$

$$\varepsilon^* = \min_\zeta \sup_\alpha \{\rho(\alpha; N^*(\cdot)) - \langle \alpha, \zeta \rangle - \varphi^*(\alpha)\} \tag{3.9}$$

In particular, if $\varphi(\chi) = \| \chi \|$, from (3.7) we obtain

$$\varepsilon^* = \min_\zeta \max_\alpha \{\rho(\alpha; N^*(\cdot)) - \langle \alpha, \zeta \rangle\}, \quad \zeta \in R^{(p)}, \| \alpha \|^* \leqslant 1 \tag{3.10}$$

Summing up what we have said, we obtain the following assertion.

Theorem 3.1. Let $\mu(\cdot)$, $\nu(\cdot)$ be given functions. The solution of deterministic problem (1.7) is obtained by the operation $\psi(z^*(\cdot),\ z^*(\cdot)) = \eta^\circ(N^*(\cdot))$, where η° is an element of functional (3.9) and (3.10), extremal with respect to ζ, computable from realization $z^*(\cdot)$ in accordance with formulas (3.7) – (3.10), (3.4). The number ε^* is the estimate, unimprovable with respect to the functional $\varphi(\chi) = \|\chi\|$, of the deviation $\chi = \eta(\vartheta) - \eta^\circ$ of the realized value $\eta(\vartheta)$ from η°.

Corollary 3.2. Let $\eta(\vartheta) = n'x(\vartheta)$ be a scalar quantity and let $\varphi(\chi) = |\chi|$. Then

$$\eta^\circ = {}^1\!/_2\,[\rho(1;\ N^*(\cdot)) - \rho(-1;\ N^*(\cdot))] \tag{3.11}$$

In fact, if N is an n'-vector, then the set $N^*(\cdot)$ is a segment whose endpoints are the numbers $a = -\rho(-1;\ N^*(\cdot))$, $b = \rho(1;\ N^*(\cdot))$. Then from (3.10) we find $\varepsilon^* = (b-a)/2 = \min_\eta \max\{b-\eta,\ -a+\eta\}$. Here the minimum is reached when $b - \eta = \eta - a$, i.e. $\eta^\circ = (a+b)/2$. The right-hand side of (3.11) is now obtained by formula (3.8).

Let us assume the functions $\mu(\cdot)$, $\nu(\cdot)$ are random. The distributions of the quantities $\mu(\cdot)$ and $\nu(\cdot)$ are now *a posteriori* and depend on $z^*(\cdot)$. Without dwelling on the computation of these distributions, we note that solution (1.6) is determined by the formula

$$\varepsilon^* = \min_\zeta M\,[\{\max_\eta \varphi(\eta - \zeta);\quad \eta \in N^*(\cdot)\} \,/\, z = z^*(\cdot)] \qquad \zeta \in R^{(p)} \tag{3.12}$$

i.e.

$$\varepsilon^* = \min_\zeta M\,[\sup_\alpha(\rho(\alpha;\quad N^*(\cdot)) - \langle\alpha,\quad \zeta\rangle - \varphi^*(\alpha)) \,/\, z = z^*(\cdot)]$$

and $\psi^\circ(z^*(\cdot),\ z^*(\cdot)) = \eta^\circ(z^*(\cdot))$, where $\eta^\circ(z^*(\cdot))$ is an element of (3.12), extremal with respect to $\zeta \in R^{(p)}$.

Theorem 3.2. Let $\mu(\cdot)$, $\nu(\cdot)$ be random processes whose *a posteriori* distributions for given $z^*(\cdot)$ are known. Then the operation $\psi^\circ(z^*(\cdot), z^*(\cdot))$, solving problem (1.5), is determined by the equality $\psi^\circ(z^*(\cdot), z^*(\cdot)) = \eta^\circ(z^*(\cdot))$, where $\eta^\circ(z^*(\cdot))$ is an element of (3.11), extremal with respect to ζ. The optimal error ε^* is found from formulas (3.11), (3.8), (3.6), (3.4).

Corollary 3.3. Let $\varphi(\chi) = \|\chi\|$. Then condition (3.12) takes the form

$$\varepsilon^* = \min_\zeta M\,[\sup_\alpha(\rho(\alpha;\ N^*(\cdot)) - \langle\alpha,\quad\zeta\rangle) \,/\, z = z^*(\cdot)] \tag{3.13}$$

$$\zeta \in R^{(p)}, \qquad \|\alpha\|^* \leqslant 1$$

Corollary 3.4. Let $\eta(\vartheta) = n'x(\vartheta)$ be a scalar quantity and let $\varphi(\chi) = |\chi|$. Then

$$\psi^\circ(z^*(\cdot),\quad z^*(\cdot)) = {}^1\!/_2 M\ [\{\rho(n;\quad X(z^*(\cdot)),\quad \mu(\cdot),\ \nu(\cdot))) - \rho(-n;\ X(z^*(\cdot)),\ \mu(\cdot),\ \nu(\cdot)))\} \,/\, z = z^*(\cdot)]$$

$$\varepsilon^* = {}^1\!/_2 M\,[\{\rho(n;\ X(z^*(\cdot)),\ \mu(\cdot),\ \nu(\cdot))) - \rho(-n;\ X(z^*(\cdot)),\ \mu(\cdot),\ \nu(\cdot)))\} \,/\, z = z^*(\cdot)]$$

In fact, the vectors $\{a(\xi) = \rho(-1;\ N^*(\cdot)),\quad b(\xi) = \rho(1;\ N^*(\cdot))\}$ be random with an *a posteriori* distribution function $F(\xi)$. By formula (3.13) we obtain

$$\varepsilon^* = \min_\eta \int_{-\infty}^{\infty} [\max\{b(\xi) - \eta,\ \eta - a(\xi)\}]\, dF(\xi), \qquad \eta \in R^{(1)}$$

We denote
$$b^* = \int_{-\infty}^{\infty} b(\xi)\,dF(\xi), \quad a^* = \int_{-\infty}^{\infty} a(\xi)\,dF(\xi)$$

The corollary's assertion then follows from the trivial inequality
$$\int_{-\infty}^{\infty} \left[\max\{b(\xi)-\eta,\ \eta - a(\xi)\} - \frac{b(\xi)-a(\xi)}{2}\right] dF(\xi) \geqslant 0$$
valid for any η.

4. A posteriori estimate by bounded operations. Let us consider system (1.1), (1.2) and problem (1.7) by now assuming that the $(m \times p)$-matrix functions $w(\cdot\ /\ z^*(\cdot)) \in W$, where W is a convex set in the $(m \times p)$-vector space B^* (adjoint to Banach space B), is closed in the weak*-topology. We assume the functions $\mu(\cdot)$, $\nu(\cdot)$ as fixed. At first let there be given an $(m \times p)$-matrix function $w(\cdot) \in W \cap \Lambda(l) = W(l)$, where l is an observable direction. From Eq.(3.1) we obtain the equality
$$\langle T_1^* w(\cdot),\ x\rangle = \langle T_2^* w(\cdot),\ v(\cdot)\rangle + \langle T_3^* w(\cdot),\ \xi(\cdot)\rangle - \langle w(\cdot),\ z^*(\cdot)\rangle \tag{4.1}$$
valid for any $\{x,\ v(\cdot),\ \xi(\cdot)\}$ consistent with $z^*(\cdot)$, i.e.
$$x \in X(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot)), \quad v(\cdot) \in V(\mu(\cdot)), \quad \xi(\cdot) \in \Xi(\nu(\cdot))$$
As a result we have
$$\langle l,\ x\rangle \leqslant \rho(T_2^* w(\cdot);\ V(\mu(\cdot))) + \rho(T_3^* w(\cdot);\ \Xi(\nu(\cdot))) - \langle w(\cdot),\ z^*(\cdot)\rangle = \Phi_+(w(\cdot),\ \mu(\cdot),\ \nu(\cdot),\ z^*(\cdot))$$

Hence we find
$$\langle l,\ x\rangle \leqslant \inf_w \Phi_+(w(\cdot),\ \mu(\cdot), \nu(\cdot),\ z^*(\cdot)), \quad w(\cdot) \in W(l) \tag{4.2}$$
and, analogously,
$$\langle l,\ x\rangle \geqslant \sup_{w(\cdot)} \Phi_-(w(\cdot),\ \mu(\cdot),\ \nu(\cdot),\ z^*(\cdot)), \quad w(\cdot) \in W(l) \tag{4.3}$$
$$\Phi_-(w(\cdot),\ \mu(\cdot),\ \nu(\cdot),\ z^*(\cdot)) = \langle w(\cdot),\ z^*(\cdot)\rangle + \rho(-T_2^* w(\cdot);\ V(\mu(\cdot))) + \rho(-T_3^* w(\cdot);\ \Xi(\nu(\cdot)))$$
i.e.
$$\langle l,\ x\rangle \geqslant -\inf_{w(\cdot)} \Phi_-(w(\cdot),\ \mu(\cdot),\ \nu(\cdot),\ z^*(\cdot)), \quad w(\cdot) \in W(l)$$

Suppose that set $W(l)$ is bounded. Then the operation $\inf$ in conditions (4.2), (4.3) can be replaced by $\min$. It is important to stress that here we do not need to know the set $W^*(z^*(\cdot),\ \mu(\cdot),\ \nu(\cdot))$ in advance in order to obtain the a posteriori estimate, because in (4.2) the estimates are realized with respect to the noise $v(\cdot) \in V(\mu(\cdot))$ and $\xi(\cdot) \in \Xi(\nu(\cdot))$.

5. Comparison of a priori and a posteriori estimates. We first consider the purely deterministic case. Thus, let $\mu(\cdot)$, $\nu(\cdot)$ be nonrandom. We compare the numbers $\varepsilon^\circ = \min \varepsilon(\mu(\cdot),\ \nu(\cdot),\ w(\cdot))$ over $w(\cdot) \in W_H$ and $\varepsilon^* = \varepsilon^*(z^*(\cdot)) = \min \varepsilon^*(\mu(\cdot),\ \nu(\cdot),\ w(\cdot\ /\ z^*(\cdot)))$ over $w(\cdot\ /\ z^*(\cdot)) \in W_H$, namely, the observation errors obtained by a priori and by a posteriori observations, respectively. Keeping the expression for $f(\mu(\cdot),\ \nu(\cdot),\ w(\cdot),\ \alpha)$, formulas (2.8), (2.9), (3.4), (3.6), and Lemma 3.4 in mind, we conclude

$$\varepsilon^\circ = \inf_{w \in W_H} \sup_\alpha \left\{ \int_{t_0}^{\vartheta} \nu(t)\rho(q(t, \alpha w(\cdot)) H(t); Q)\, dt + \right. \tag{5.1}$$
$$\left. \int_{t_0}^{\vartheta} \mu(t)\rho(s(t, \alpha w(\cdot)) B(t) - q(t, \ \alpha w(\cdot)) C(t); P)\, dt - \varphi^*(\alpha) \right\} \geqslant$$
$$\sup_\alpha \{\rho(\alpha N; X(0, \mu(\cdot), \nu(\cdot))) - \varphi^*(\alpha)\} = \varepsilon^*(0), \quad \alpha \in R^{(p)}$$

Note that the set $X(0, \mu(\cdot), \nu(\cdot))$ is symmetric with respect to the origin (this follows from the corresponding definition and symmetry of sets P, Q). Therefore, (5.1) admits the following interpretation.

Lemma 5.1. Let $\mu(\cdot), \nu(\cdot)$ be given deterministic functions. The error ε° of an a priori observation of parameter $\eta(\vartheta)$ with respect to signal $y(\cdot)$ in (1.1), (1.2) is not less than $\varepsilon^*(0)$, the unimprovable error estimate for an a posteriori observation of parameter $\eta(\vartheta)$ with respect to signal $z^*(\cdot) \equiv 0$ in (1.1), (1.2).

By direct calculation we convince ourselves that if α is a scalar and $\varphi(\chi) = |\chi|$, then $\varepsilon^\circ = \varepsilon^*(0)$.

We arrive at the next result by taking into account that the error ε° of a priori observation is achieved by a linear operation $\langle w(\cdot), z(\cdot)\rangle$, and that, on the basis of Lemma 5.1, it cannot be improved.

Corollary 5.1. If α is a scalar and $\varphi(\chi) = |\chi|$, then the unimprovable a priori estimate is achieved by the linear operation $\langle w(\cdot), z(\cdot)\rangle$ satisfying conditions (2.6), (2.7) and the moment equalities (2.3).

Let us go on to compare $\varepsilon^*(0)$ and $\varepsilon^*(z^*(\cdot))$, to do this we compare $W^*(0, \mu(\cdot), \nu(\cdot))$ with $W(z^*(\cdot), \mu(\cdot), \nu(\cdot))$. We denote the m-dimensional space spanned by the row-vectors of matrix $Z(t, \vartheta)$ by the symbol H^m. ($H^m = \{h(t): h(t) = Z(t, \vartheta)\, l$ for some $l \in R^{(n)}\}$). Then from formula (3.1) we conclude

$$W^*(0, \ \mu(\cdot), \ \nu(\cdot)) = \{v(\cdot), \ \xi(\cdot): \ T_2 v(\cdot) + T_3 \xi(\cdot) \in H^m\} \tag{5.2}$$
$$v(\cdot) \in V(\mu(\cdot)), \qquad \xi(\cdot) \in \Xi(\nu(\cdot))$$

and, analogously, un general form, for these same classes

$$W^*(z^*(\cdot), \ \mu(\cdot), \ \nu(\cdot)) = \{v(\cdot), \ \xi(\cdot): \ T_2 v(\cdot) + T_3 \xi(\cdot) - z^*(\cdot) \in H^m\} \tag{5.3}$$

We shall treat the functions $z^*(\cdot)$ as elements of space $L_2^{(m)}$, i.e. we shall assume $B = L_2^{(m)}$ (see Sect. 3). Then, also $H^m \in L_2^{(m)}$. Let H_1^m denote the orthogonal complement of H^m in $L_2^{(m)}$. Any element $h(\cdot) \in L_2^{(m)}$ can now be represented in the form $h(\cdot) = (h(\cdot))_0 + (h(\cdot))_1$, where $(h(\cdot))_0 \in H^m$, $(h(\cdot))_1 \in H_1^m$. It can be verified that any pair $\{v(\cdot), \xi(\cdot)\} \in W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ is representable in the form

$$v(\cdot) = v^\circ(\cdot) + v^*(\cdot), \qquad \xi(\cdot) = \xi^\circ(\cdot) + \xi^*(\cdot) \tag{5.4}$$

where the element $\{v^\circ(\cdot), \xi^\circ(\cdot)\} \in W^*(0, \mu(\cdot), \nu(\cdot))$ depends, in general, on $\{v(\cdot), \xi(\cdot)\}$, and the element $\{v^*(\cdot), \xi^*(\cdot)\} \in W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ is fixed (i.e. does not now depend on $\{v(\cdot), \xi(\cdot)\}$). Then

$$W^*(z^*(\cdot), \ \mu(\cdot), \ \nu(\cdot)) = W^\circ(0, \ \mu(\cdot), \ \nu(\cdot) / z^*(\cdot)) + w^*(z^*(\cdot), \ \mu(\cdot), \ \nu(\cdot)) \tag{5.5}$$

Here $W^\circ(0, \mu(\cdot), \nu(\cdot) / z^*(\cdot)) = \{\{v^\circ(\cdot), \xi^\circ(\cdot)\}\}$ is the set of elements $\{v^\circ(\cdot), \xi^\circ(\cdot)\}$ obtained from (5.4) and $w^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)) = \{v^*(\cdot), \xi^*(\cdot)\}$. It is clear that $W^\circ(0, \mu(\cdot), \nu(\cdot) / z^*(\cdot)) \subset W^*(0, \mu(\cdot), \nu(\cdot))$ We denote

$$T_2 v^*(\cdot) + T_3 \xi^*(\cdot) = f^*(\cdot), \qquad f^*(\cdot) = (f^*(\cdot))_0 + (f^*(\cdot))_1 \tag{5.6}$$

Taking the structure of set $W^\circ(0, \mu(\cdot), \nu(\cdot) / z^*(\cdot))$, element $f^*(\cdot)$ in (5.6), and formula (3.6) in Lemma 3.9 into account, by direct calculations we convince ourselves of the validity of the equality

$$\rho(l; X(z^*(\cdot), \mu(\cdot), \nu(\cdot))) = \langle \lambda^\circ(\cdot), (z^*(\cdot) + f^*(\cdot))_0\rangle + \rho(s(t, \lambda^\circ(\cdot)) B(t) - q(t, \lambda^\circ(\cdot)) C(t), q(t, \lambda^\circ(\cdot)) H(t); W^\circ(0, \mu(\cdot), \nu(\cdot) / z^*(\cdot))) \tag{5.7}$$

Here $\lambda^\circ(\cdot) \in H^m$ is the unique solution of the equation $T_1^* \lambda(\cdot) = l$ under the condition $\langle \lambda^\circ(\cdot), \lambda^\circ(\cdot)\rangle = \min$. From formulas (5.7), (3.8), (3.9), (2.6) we conclude $(\zeta, \alpha \in R^{(p)})$

$$\varepsilon^*(z^*(\cdot)) = \min_\zeta \sup_\alpha \{\rho(\alpha N; X(z^*(\cdot), \mu(\cdot), \nu(\cdot))) - \langle \alpha, \zeta\rangle - \varphi^*(\alpha)\} \leqslant \sup_\alpha \{\rho(s(t; \overset{\circ}{\lambda}_{\alpha N}(\cdot)) B(t) - q(t, \overset{\circ}{\lambda}_{\alpha N}(\cdot)) \cdot C(t) \; q(t, \overset{\circ}{\lambda}_{\alpha N}(\cdot)) H(t); W^\circ(0, \mu(\cdot), \nu(\cdot) / z^*(\cdot)) - \varphi^*(\alpha)\} \leqslant \varepsilon^*(0) \tag{5.8}$$

Inequality (5.8) and Lemma 5.1 lead to the following assertion.

T h e o r e m 5.1. Let $\mu(\cdot)$, $\nu(\cdot)$ be given deterministic functions. Then the error ε° of a p r i o r i observation of parameter $\eta(\vartheta)$ with respect to any signal $y(\cdot)$ in (1.1), (1.2) is not less than the number $\varepsilon^*(z^*(\cdot))$, the unimprovable estimate of the error of a p o s t e r i o r i observation of parameter $\eta(\vartheta)$ with respect to signal $y^*(\cdot)$.

N o t e 5.1. If signal $z^*(\cdot) \in H^m$, then the set $w^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)$ contains a zero element. It then follows from (5.2), (5.3) that $W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot)) = W^*(0, \mu(\cdot), \nu(\cdot))$. Thus, the observation of signal $z^*(\cdot) \in H^m$ does not give any additional information, permitting us to lessen the error $\varepsilon^*(z^*(\cdot))$ in comparison with the estimate $\varepsilon^*(0)$ (and when $\varepsilon^*(0) = \varepsilon^\circ$, even in comparison with the estimate ε° of a p r i o r i observation).

We now show that the unimprovable operation $\psi(z^*(\cdot), z^*(\cdot))$ realizing the "φ-center" of set $N^*(\cdot)$ (see Sect. 3) is achieved by operations of the form

$$\psi(z^*(\cdot), z^*(\cdot)) = \langle w(\cdot / z^*(\cdot)), z^*(\cdot)\rangle \tag{5.9}$$

In fact, the desired operation must satisfy the following moment equalities:

$$\langle w(\cdot / z^*(\cdot)), z^*(\cdot)\rangle = \eta^\circ, \; \langle w(\cdot / z^*(\cdot)), Z(\cdot, \vartheta)\rangle = -N \tag{5.10}$$

If the rows n_i' of matrix N are observable directions and $(z^*(\cdot))_1 \neq 0$, then on the basis of known results in control theory [1] we conclude that problem (5.10) is solvable in B^* for any η°, N.

Let $z^*(\cdot) \in H^m$, i.e. $z^*(t) = Z(t, \vartheta) c$, where $c \in R^m$. The set $W^*(z^*(\cdot), \mu(\cdot), \nu(\cdot))$ is then symmetric with respect to the origin. Hence by a direct calculation we are convinced that the set $N^*(\cdot) = NX(z^*(\cdot), \mu(\cdot), \nu(\cdot)) \subset R^{(p)}$ is symmetric with respect to the point $\eta^\circ = Nc$, which, in the given case, also gives the

"Chebyshev center" of set $N^*(\cdot)$. Consequently, any operation $w(\cdot / z^*(\cdot))$ satisfying the non-bias condition (5.10) gives us, under this condition, the unimprovable estimate of $\eta^\circ(N^*(\cdot))$. We note that the unimprovable estimate of $\eta^\circ(N^*(\cdot))$ is achieved in the general case by operation (5.9) which, generally speaking, is not linear in $z^*(\cdot) \in B$.

Now let the constraints $\mu(\cdot)$, $\nu(\cdot)$ be probabilistic processes with known a priori distributions. Then from (3.11), (1.3) we have

$$\varepsilon^*(z^*(\cdot)) = \min_\zeta M\{\sup_\alpha[\rho(\alpha;\ N^*(\cdot)) - \langle\alpha,\ \zeta\rangle - \varphi^*(\alpha)]/ z = z^*(\cdot)\} \leqslant \min_\zeta \max_{y(\cdot)} M\{\sup_\alpha[\rho(\alpha; N^*(\cdot)) - \langle\alpha,\ \zeta\rangle - \varphi^*(\alpha)] / y = y(\cdot)\} \leqslant \varepsilon^\circ \tag{5.11}$$

The validity of the following assertion ensues from this and from Theorem 5.1.

Theorem 5.2. Let $\mu(\cdot)$, $\nu(\cdot)$ be given random processes. Then the error (3.11) in the problem of a posteriori observation of the vector-valued parameter $\eta(\vartheta)$ with respect to any signal $y^*(\cdot)$ in (1.1), (1.2) does not exceed the error ε° in the corresponding a priori observation problem.

We note that the consideration of the a posteriori observation operations is important for describing problems of conflicting controls with incomplete information on the system's position, which are close to the problems studied in monograph [12].

Note 5.2. The mean in (5.11) is computed with respect to the a posteriori distribution of $(\mu(\cdot),\ \nu(\cdot))$ [13]. The distribution is constructed in such a way that realizations $\mu(\cdot)$, $\nu(\cdot)$ for which the signal $z^*(\cdot)$ cannot be observed are excluded.

Note 5.3. In this paper we have assumed that a continuous measurement of signal $z^*(\cdot)$ is possible; however, the arguments go through also for the more general linear "measurement operators" $Mx(\cdot) = z^*(\cdot)$. The investigations in [14, 15] are devoted to the optimal choice of the method of measuring the signal $x(\cdot)$.

6. Examples. 1) Let us consider the problem of observing the quantity $\eta = x_2(\vartheta)$ relative to the system

$$x_1^\cdot = x_2,\quad x_2^\cdot = v \tag{6.1}$$

from a signal $y(t)$, $t \in [0, \vartheta]$, connected with system (6.1) by the equation

$$y^\cdot = x_2 + v \tag{6.2}$$

Here the noise $v(t)$ is constrained by $|v(t)| \leqslant \mu$ (μ is a random quantity with an a priori distribution $F(u)$) concentrated on $[0, \mu_0]$.

We first estimate the error ε° of a priori observation. This is sufficient, according to the results in Sect. 5, to consider the signal $y^*(t) \equiv 0$ and to compute the quantity $\varepsilon^*(0, \mu) = \varepsilon^\circ(\mu)$. From (6.2) it follows that the set $W^*(0, \mu)$ (for fixed μ) consists of the solutions of the equation $v^\cdot + v = 0$, constrained by the condition $|v| \leqslant \mu$. Thus, $W^*(0, \mu) = \{v(t) : v(t) = ce^{-t},\ |c| \leqslant \mu\}$. For the quantity $x_2(\vartheta)$ we then have the estimate

$$\min_{v(\cdot)\in W^*(0,\mu)}[-v(\vartheta)] \leqslant x_2(\vartheta) \leqslant \max_{v(\cdot)\in W^*(0,\mu)}[-v(\vartheta)]$$

or $\mu e^{-\vartheta} \leqslant x_2(\vartheta) \leqslant \mu e^{-\vartheta}$, consequently,

$$\varepsilon^\circ(\mu) = \mu e^{-\vartheta} = \varepsilon^*(0,\ \mu)$$

If μ is a random quantity, then the a priori estimate $\varepsilon^\circ = \max M\{\varepsilon^\circ(\mu) / y = y(\cdot)\}$ with respect to $y(\cdot)$. This maximum is achieved, for example, by the signal $y(t) = \alpha t^2/2$.

where $\alpha = \mu_0 \operatorname{cth}(\vartheta / 2)$, because for such a signal the conditional distribution degenerates and yields the constant μ_0.

2) Let us consider problem (6.1), (6.2) of a posteriori observation of the quantity $\eta = x_2(\vartheta)$ with respect to the signal $y^*(t) = t^2 / 2$. The set $W^*(y^*(\cdot), \mu)$ (μ is fixed) is now determined by the conditions

$$v(t) = ce^{-t} + 1, \ -(1+\mu) \leqslant c \leqslant (\mu - 1) e^{\vartheta} \quad \text{for} \quad \mu^* \leqslant \mu \leqslant \mu_0$$

The set $W(y^*(\cdot), \mu)$ is empty for $\mu < \mu^* = \operatorname{th}(\vartheta / 2)$. For the estimate η of quantity $x_2(\vartheta)$ we have

$$\eta(y^*(\cdot), \mu) = \vartheta + (\mu + 1)(e^{-\vartheta} + 1) / 2 \quad \text{for} \ \mu^* \leqslant \mu \leqslant \mu_0$$

The error of the estimate is determined by the equality

$$\varepsilon^*(y^*(\cdot), \mu) = (1 + e^{-\vartheta})(\mu - \mu^*) / 2 \quad \text{for} \ \mu^* \leqslant \mu \leqslant \mu_0$$

We immediately verify that $\varepsilon^*(y^*(\cdot), \mu) < \varepsilon^*(0, \mu)$. If μ is a random quantity, the error ε^* is determined by the relations

$$\varepsilon^* = M\left\{\varepsilon^*(y^*(\cdot), \mu) / y^*(t) = \frac{t^2}{2}\right\} = \int_{\mu^*}^{\mu_0} \beta(u - \mu^*)\, dF\left(u / y^*(t) = \frac{t^2}{2}\right) =$$

$$p^{-1} \int_{\mu^*}^{\mu_0} \beta(u - \mu^*)\, dF(u) \leqslant \beta(\mu_0 - \mu^*) < \varepsilon^\circ$$

$$\beta = \frac{1 + e^{-\vartheta}}{2}, \qquad p = \int_{\mu^*}^{\mu_0} dF(u)$$

Here $F(u / y^* = t^2 / 2)$ is the a posteriori distribution of quantity μ.

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Translated by N. H. C.

UDC 62-50

ON A SUFFICIENT ENCOUNTER CONDITION IN A DIFFERENTIAL GAME

PMM Vol. 37, №5, 1973, pp. 787-793
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(Received April 3, 1973)

We consider a position differential game of encounter with a target set at a specified instant. We derive one sufficient condition by whose fulfillment the pursuer ensures himself a definite qualitative result of the game. The construction of the first player's strategy is based on the program construction introduced in [1 – 3]. The results abut the investigations in [1 – 5].

1. We consider a conflict-controlled system described by the vector differential equation

$$dx/dt = A\ (t)x + f\ (t,\ u,\ v) \qquad (1.1)$$
$$x\ [t_0] = x_0, \quad u \in P, \quad v \in Q$$

Here $f\ (t,\ u,\ v)$ is a continuous n-dimensional vector-valued function, u and v are the player's controls, P and Q are compacta in appropriate vector spaces. By $\{x\}_m$ we denote the vector composed from the first m $(m \leqslant n)$ coordinates of vector x. By the problem's hypothesis a convex bounded closed set M is given in the space $\{x\}_m$. The first player, directing the choice of control u, strives to encounter this set by an instant ϑ known in advance. The second player (v) obstructs this.

Let us refine the problem statement. By the first player's position strategy $U = U\ (t,\ x)$ we mean a mapping which associates a set $U\ (t,\ x) \subset P$ with each game position $\{t,\ x\}$. Any absolutely continuous function $x\ [t] = x\ [t;\ t_0, x_0, U]$, being a uniform limit of the Euler polygonal lines $x_\Delta\ [t] = x_\Delta\ [t;\ t_0,\ x_0,\ U]$ which satisfy the following condition

$$\frac{dx_\Delta}{dt} \in A\ (t)\, x_\Delta + F\ (t, u\ [\tau_i]) \qquad (1.2)$$
$$x_\Delta\ [t_0] = x_0$$

is called a motion of system (1.1) generated by strategy U. Here